Laplace Transformer (拉普拉斯轉換；拉氏轉換)(電路學&工數)

一、Definition：$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$, $s>0$

二、基本函數：

步階 (1) $\mathcal{L}[u(t)] = \frac{1}{s}$

$\mathcal{L}[u(t-a)] = \frac{1}{s} e^{-as}$

冪函數 (2) $\mathcal{L}[t^n]$, $n \in N = \frac{n!}{s^{n+1}}$

指數 (3) $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $s>a$

三角 (4) $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$

$\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}$

脈衝 (5) $\mathcal{L}[\delta(t)] = 1$; $\mathcal{L}[\delta(t-a)] = e^{-as}$

斜波 (6) $\mathcal{L}[t] = \frac{1}{s^2}$

三、運算法則：

先移S再移T

(1) S domain shift (S變數平移)

$\mathcal{L}[e^{at} f(t)] = F(s-a)$

(2) Time domain shift (時間平移)：$\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

微X積÷

(3) 微分後的 Laplace Transformer (用在L和C上)

$$\begin{cases} \mathcal{L}[f'(t)] = sF(s) - f(0) \\ \mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0) \end{cases}$$

(4) 積分後的 Laplace Transformer

$$\begin{cases} \mathcal{L}[\int_0^t f(z)dz] = \frac{1}{s} F(s) \\ \mathcal{L}[\int_0^t \int_0^y f(z)dz\,dy] = \frac{1}{s^2} F(s) \end{cases}$$

(5) L.T. 後的微分：

$$\begin{cases} (1)\ \frac{d}{ds} F(s) = -\mathcal{L}[tf(t)] \\ (2)\ \frac{d}{ds^2} F(s) = (-1)^2 \mathcal{L}[t^2 f(t)] \end{cases}$$

(6) L.T. 後的積分

$$\begin{cases} (1)\ \int_s^\infty F(z)dz = \mathcal{L}[\frac{1}{t} f(t)] \\ (2)\ \int_s^\infty \int_u^\infty F(z)dz\,du = \mathcal{L}[\frac{1}{t^2} f(t)] \end{cases}$$

(7) 週期函數

$\because f(t+p) = f(t)$ (P為f(t)之週期)

$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p f(t) e^{-st} dt$

(8) 初值定理：$f(0) = \lim_{s\to\infty} sF(s)$

(9) 終值定理：$f(\infty) = \lim_{s\to 0} sF(s)$ → 條件：分母根實部需為負

(10) Convolution theorem (摺積；卷積；迴旋定理)

$f(t) * g(t) = \int_0^t f(t-\alpha) g(\alpha) d\alpha$

$Y(s) = F(s) G(s) = \mathcal{L}[f(t) * g(t)]$

Laplace 逆轉換 (Inverse Laplace Transformer)

一、基本型：

(1) $\mathcal{L}^{-1}\left[\frac{1}{s}\right]=1$

(2) $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$

(3) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

(4) $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

(5) $\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$

二、S domain shift Inverse L.T.

$$\mathcal{L}[e^{at}f(t)]=F(s-a) \Rightarrow \mathcal{L}^{-1}[F(s-a)]=e^{at}f(t)=e^{at}\mathcal{L}^{-1}[F(s)]$$

三、t domain shift Inverse L.T.

$$\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s) \Rightarrow \mathcal{L}^{-1}[e^{-as}F(s)]=f(t-a)u(t-a)$$

四、微分後的 Inverse L.T.：

$$\begin{cases} \frac{d}{ds}F(s)=-\mathcal{L}[tf(t)] \Rightarrow \mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right]=-tf(t) \Rightarrow ※\mathcal{L}^{-1}\left[\frac{1}{[f(s)]^2}\right] \Rightarrow \overset{ex:}{}\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] \\ \frac{d^2}{ds^2}F(s)=\mathcal{L}[t^2f(t)] \Rightarrow \mathcal{L}^{-1}\left[\frac{d^2}{ds^2}F(s)\right]=t^2f(t) \end{cases}$$

五、積分後的 Inverse L.T.：

$$\begin{cases} \frac{1}{s}F(s)=\mathcal{L}\left[\int_0^t f(z)\,dz\right] \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t f(z)\,dz \\ \frac{1}{s^2}F(s)=\mathcal{L}\left[\int_0^t\int_0^y f(z)\,dz\,du\right] \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2}F(s)\right]=\int_0^t\int_0^y f(z)\,dz\,du \end{cases}$$

第五章 Laplace (拉普拉斯) Transformer (工數版)

§5-1 基本概念

一、意義 $f(t) \xrightarrow{\text{Laplace 轉換}} F(s) \xrightarrow{\text{Laplace 逆轉換}} f(t)$

滿足2個條件：①收斂 ②有初始值或邊際值

二、Laplace define：

$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s)$，$s>0$ (瑕積分)

三、Laplace transformer 存在性定理

$\forall N>0$，$f(t)$ 在 $[0,N]$ 分段 continue，$\exists p>0$，且 $s>Q>0$

$\ni t>N$ 時，$|f(t)| \le pe^{at}$，即 $\mathcal{L}[f(t)]$ 存在

$\boxed{例}$ (1) $f(t)=\frac{1}{\sqrt{t}}$ (2) $f(t)=e^{t^2}$，求 $\mathcal{L}[f(t)]$ 是否存在？

<sol> (1) $\lim\limits_{t\to\infty} f(t)e^{-st} = \lim\limits_{t\to\infty} \frac{\frac{1}{\sqrt{t}}}{e^{st}} = 0$ $\therefore \mathcal{L}[f(t)]$ 存在

(2) $\lim\limits_{t\to\infty} f(t)e^{-st} = \lim\limits_{t\to\infty} \frac{e^{t^2}}{e^{st}} = \infty$ $\therefore \mathcal{L}[f(t)]$ 不存在

§5-2 六大基本函數 laplase transformer

一、單位步階 (unit-step) (為了解某一段函數所使用)

(一) 定義

(1) $u(t)=\begin{cases}1 & ; t\ge 0\\ 0 & ; t<0\end{cases}$ (2) $u(t-a)=\begin{cases}1 & ; t\ge a\\ 0 & ; t<a\end{cases}$

結論：$f(t-a)u(t-a)$ 即 $f(t)$ 往右平移 a，取 $x>a$ 的部分

(二) Lplace Transformer

(1) $\mathcal{L}[u(t)] = \frac{1}{s}$ (2) $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$

<pf> (1) $\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = 0-(-\frac{1}{s}) = \frac{1}{s}$

(2) $\mathcal{L}[u(t-a)] = \int_0^\infty u(t-a)e^{-st}dt = \int_a^\infty 1\cdot e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_a^\infty = 0-(-\frac{1}{s}e^{-as}) = \frac{1}{s}e^{-as}$

$\boxed{例}$ $f(t)=\begin{cases}0 & , 0\le t<1\\ 1 & , 1\le t<3\\ -1 & , t\ge 3\end{cases}$ find $\mathcal{L}[f(t)]=?$

<sol> $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_1^3 1\cdot e^{-st}dt + \int_3^\infty -1\cdot e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_1^3 + \frac{1}{s}e^{-st}\Big|_3^\infty$

$= -\frac{1}{s}e^{-3s} + \frac{1}{s}e^{-s} + 0 - \frac{1}{s}e^{-3s} = \frac{1}{s}e^{-s} - \frac{2}{s}e^{-3s}$ #

二、冪函數

(1) $f(t)=t^n$，$n\in N \Rightarrow \mathcal{L}[f(t)] = \frac{n!}{s^{n+1}}$

(2) $f(t)=t^a$，$a>0 \Rightarrow \mathcal{L}[f(t)] = \frac{\Gamma(a+1)}{s^{n+1}}$ ※ Γ (gama)

<pf> $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty t^n e^{-st}dt = \frac{1}{s^{n+1}}\int_0^\infty (st)^n e^{-st}d(st) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

EX: $\mathcal{L}[t^5] = \frac{5!}{s^6}$ $\mathcal{L}[t^{\frac{3}{2}}] = \frac{\Gamma\frac{5}{2}}{s^{\frac{5}{2}}} = \frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}}$

三、指數函數

$f(t)=e^{at} \Rightarrow \mathcal{L}[f(t)] = \frac{1}{s-a}$，$s>a$

<pf> $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = 0-\left[\frac{1}{-(s-a)}\right]$

$= \frac{1}{s-a}$

EX: $\mathcal{L}[e^{3t}] = \frac{1}{s-3}$

四、三角函數

(1) $f(t)=\cos at \Rightarrow \mathcal{L}[f(t)]=\dfrac{s}{s^2+a^2}$

(2) $f(t)=\sin at \Rightarrow \mathcal{L}[f(t)]=\dfrac{a}{s^2+a^2}$

分部積分

<p.f.> (1) $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty(\cos at)e^{-st}dt=\dfrac{e^{-st}}{s^2+a^2}[-s\cos at+a\sin at]\Big|_0^\infty$

$=\dfrac{s}{s^2+a^2}$

EX: $\mathcal{L}[\cos 3t]=\dfrac{s}{s^2+3^2}$, $\mathcal{L}[\sin 3t]=\dfrac{3}{s^2+3^2}$

例: $\mathcal{L}[\cos(\omega t+\phi)]=\mathcal{L}[\cos\omega t\cdot\cos\phi-\sin\omega t\cdot\sin\phi]$

$=\cos\phi\,\mathcal{L}[\cos\omega t]-\sin\phi\cdot\mathcal{L}[\sin\omega t]$

$=\cos\phi\cdot\dfrac{s}{s^2+\omega^2}-\sin\phi\cdot\dfrac{\omega}{s^2+\omega^2}=\dfrac{s\cos\phi-\omega\sin\phi}{s^2+\omega^2}$

例: $f(t)=\sin^2(\frac{\pi t}{2})$

<sol>: $\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}\Rightarrow\sin^2\frac{\pi t}{2}=\left(\pm\sqrt{\frac{1-\cos\pi t}{2}}\right)^2=\frac{1}{2}(1-\cos\pi t)$

$\mathcal{L}[f(t)]=\mathcal{L}[\frac{1}{2}(1-\cos\pi t)]=\frac{1}{2}\mathcal{L}[1-\cos\pi t]=\frac{1}{2}(\frac{1}{s}-\frac{s}{s^2+\pi^2})$

$=\frac{1}{2}\times\dfrac{s^2+\pi^2-s^2}{s(s^2+\pi^2)}=\frac{1}{2}\times\dfrac{\pi^2}{s(s^2+\pi^2)}$

五、雙曲線

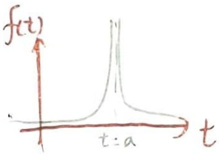
(1) $f(t)=\cosh(at)=\dfrac{e^{at}+e^{-at}}{2}\Rightarrow\mathcal{L}[f(t)]=\dfrac{s}{s^2-a^2}$

(2) $f(t)=\sinh(at)=\dfrac{e^{at}-e^{-at}}{2}\Rightarrow\mathcal{L}[f(t)]=\dfrac{a}{s^2-a^2}$

<pf> $\mathcal{L}[f(t)]=\mathcal{L}[\dfrac{e^{at}+e^{-at}}{2}]=\frac{1}{2}\{\mathcal{L}[e^{at}]+\mathcal{L}[e^{-at}]\}=\frac{1}{2}[\frac{1}{s-a}+\frac{1}{s+a}]$

$=\frac{1}{2}\times\dfrac{(s-a)+(s+a)}{(s-a)(s+a)}=\dfrac{s}{s^2-a^2}$

六、脈衝函數



(一) 定義 $\delta(t-a)=\begin{cases}\infty & t=a\\ 0 & t\neq a\end{cases}$

且 $\int_{-\infty}^{\infty}\delta(t-a)dt=1$

(二) Laplace Transformer

$\mathcal{L}[\delta(t-a)]=e^{-as}$

<pf> $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty\delta(t-a)e^{-st}dt=\int_0^\infty\delta(t-a)e^{-sa}dt$

$=e^{-as}\underbrace{\int_0^\infty\delta(t-a)dt}_{=1}=e^{-as}$

例 14. (1) $f(t)=2e^{3t}+t^2-2\cos(2t)+5\sin(3t)$

<sol> $\mathcal{L}[f(t)]=\mathcal{L}[2e^{3t}]+\mathcal{L}[t^2]-\mathcal{L}[2\cos(2t)]+\mathcal{L}[5\sin(3t)]$

$=\dfrac{2}{s+3}+\dfrac{2!}{s^3}-\dfrac{2s}{s^2+2^2}+\dfrac{5\times3}{s^2+3^2}$

§5-3 Laplace Transformer 運算法則

一、S變數平移 (s domain shift) (第一平移定理) ※先平移s再平移t

若 $\mathcal{L}[f(t)]=F(s)$; $s>a \Rightarrow \mathcal{L}[e^{at}f(t)]=F(s-a)$

二、t變數平移 (t domain shift) (第二平移定理)

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow \mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s)$

<pf> (1) $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty e^{at}f(t)e^{-st}dt=\int_0^\infty f(t)e^{-(s-a)t}dt=F(s-a)$

ex: $\mathcal{L}[\sin 3t]=\frac{s}{s^2+3^2}$; $\mathcal{L}[e^{2t}\sin 3t]=\frac{3}{(s-2)^2+3^2}$

(2) $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\int_0^\infty f(t-a)\underline{u(t-a)}e^{-st}dt$

$=\int_a^\infty f(t-a)e^{-st}dt$ 令 $k=t-a \Rightarrow t=k+a \Rightarrow dt=dk$

$=\int_a^\infty f(k)e^{-s(k+a)}dk=\int_0^\infty f(k)e^{-sk}\cdot e^{-sa}dk=e^{-as}\int_0^\infty f(k)e^{-sk}dk$

$=e^{-as}\int_0^\infty f(t)e^{-st}dt=e^{-as}F(s)$ ※

例21: $f(t)=2u(t)-2u(t-\pi)+u(t-2\pi)\sin t$ 　　 sin t 週期 $=2\pi$

<sol> $\mathcal{L}[f(t)]=\mathcal{L}[2u(t)]-\mathcal{L}[2u(t-\pi)]+\mathcal{L}[u(t-2\pi)\sin t]$

$=2\mathcal{L}[u(t)]-2\mathcal{L}[u(t-\pi)]+\mathcal{L}[\sin(t-2\pi)u(t-2\pi)]$

$=2\frac{1}{s}-2\frac{1}{s}e^{-\pi s}+\frac{1}{s^2+1^2}e^{-2\pi s}$

$=\frac{2}{s}-\frac{2}{s}e^{-\pi s}+\frac{1}{s^2+1}e^{-2\pi s}$ ※

例20: $f(t)=t^2u(t-2)$ Find $\mathcal{L}[f(t)]=?$

<sol> $\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s)$

$f(t)=t^2=[(t-2)+2]^2=(t-2)^2+4(t-2)+4 \Rightarrow f(t)=t^2u(t-2)=[(t-2)^2+4(t-2)+4]u(t-2)$

$\because \mathcal{L}[t^2+4t+4]=\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}$

$\Rightarrow \mathcal{L}\{[(t-2)^2+4(t-2)+4]u(t-2)\}=(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s})e^{-2s}$ ※

三、微分後的 Laplace Transformer (微 × 積 ÷)

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow \begin{cases}\mathcal{L}[f'(t)]=sF(s)-f(0)\\ \mathcal{L}[f''(t)]=s^2F(s)-sf(0)-f'(0)\end{cases}$

四、積分後的 Laplace Transformer

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow \begin{cases}\mathcal{L}[\int_0^t f(z)dz]=\frac{1}{s}F(s)\\ \mathcal{L}[\int_0^t\int_0^y f(z)dzdy]=\frac{1}{s^2}F(s)\end{cases}$

<pf> $\mathcal{L}[f'(t)]=\int_0^\infty f'(t)e^{-st}dt=\int_0^\infty \underbrace{e^{-st}}_{u}\underbrace{f'(t)dt}_{dv}=\int udv=uv-\int vdu$

令 $\begin{cases}u=e^{-st} & du=-se^{-st}dt\\ dv=f'(t)dt & v=f(t)\end{cases}$

$=e^{-st}f(t)\Big|_0^\infty-\int_0^\infty f(t)(-se^{-st})dt=0-f(0)+s\underbrace{\int_0^\infty f(t)e^{-st}dt}_{F(s)}$

$=sF(s)-f(0)$

<pf> $\mathcal{L}[\int_0^t f(z)dz]=\int_0^\infty\int_0^t f(z)dz\,e^{-st}dt=\int_0^\infty\int_0^t f(t)e^{-st}dzdt=\int_0^\infty\int_z^\infty f(t)e^{-st}dtdz$

$=\int_0^\infty f(z)\frac{-e^{-st}}{-s}\Big|_z^\infty dz=\int_0^\infty f(z)\frac{1}{s}e^{-sz}dz=\frac{1}{s}\int_0^\infty f(z)e^{-sz}dz$

$=\frac{1}{s}\int_0^\infty f(t)e^{-st}dt=\frac{1}{s}\mathcal{L}[f(t)]=\frac{1}{s}F(s)$

例23 $y''+4y'+4y=2t^2 \quad y(0)=y'(0)=0 \quad Y(s)=\frac{a}{s^3}+\frac{b}{s^2}+\frac{c}{s}+\frac{d}{(s+2)^2}+\frac{e}{s+2}$ Find a~e=?

<sol> $\mathcal{L}[y(t)]=Y(s)$

$\Rightarrow \mathcal{L}[y''+4y'+4y]=\mathcal{L}[2t^2] \Rightarrow \mathcal{L}[y'']+4\mathcal{L}[y']+4\mathcal{L}[y]=\mathcal{L}[2t^2]$

$\Rightarrow [s^2Y(s)-sf(0)-f'(0)]+4[sY(s)-f(0)]+4Y(s)=2\frac{2!}{s^3}$

$\Rightarrow s^2Y(s)+4sY(s)+4Y(s)=\frac{4}{s^3} \Rightarrow (s^2+4s+4)Y(s)=\frac{4}{s^3} \Rightarrow (s+2)^2Y(s)=\frac{4}{s^3}$

$\Rightarrow Y(s)=\frac{1}{s^3(s+2)^2}=\frac{1}{s^3}+\frac{-1}{s^2}+\frac{\frac{3}{4}}{s}+\frac{-\frac{1}{2}}{(s+2)^2}+\frac{-\frac{3}{4}}{s+2}$

例24. $\mathcal{L}[\int_0^t \cos(5z)dz]$

<sol> $\mathcal{L}[\cos(5z)dz]=\frac{s}{s^2+5^2}=\frac{s}{s^2+25}=F(s) \Rightarrow \mathcal{L}[\int_0^t \cos(5z)dz]=\frac{1}{s}F(s)=\frac{1}{s}\times\frac{s}{s^2+25}=\frac{1}{s^2+25}$ #

例24. $f(t)=\int_0^t e^{-2z}\sin(4z)dz$ Find $\mathcal{L}[f(t)]$=? e^{-2t} 平移

<sol> (1)先求 $\mathcal{L}[\sin(4z)dz]=\frac{4}{s^2+4^2}=\frac{4}{s+16}$

(2)再求 $\mathcal{L}[e^{-2t}\sin 4t]=\frac{4}{s^2+16}=\frac{4}{(s+2)^2+16} \Rightarrow \mathcal{L}[\int_0^t e^{-2t}\sin 4t\,dt]=\frac{1}{s}\times\frac{4}{(s+2)^2+16}$

五 Laplace Transformer 後的微分 (微×積÷)

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow \begin{cases}(1)\ \frac{d}{ds}F(s) \Leftarrow -\mathcal{L}[tf(t)] \\ (2)\ \frac{d}{ds^2}F(s)=(-1)^2\mathcal{L}[t^2f(t)]\end{cases}$

六 Laplace Transformer 後的積分

.若 $\mathcal{L}[f(t)]=F(s) \Rightarrow \begin{cases}\int_s^\infty F(z)dz \Leftarrow \mathcal{L}[\frac{1}{t}f(t)] \\ \int_s^\infty\int_u^\infty F(z)dz\,du=\mathcal{L}[\frac{1}{t^2}f(t)]\end{cases}$

<pf> $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=F(s)$

$\Rightarrow \frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt=\int_0^\infty f(t)\frac{\partial}{\partial s}e^{-st}dt=\int_0^\infty f(t)e^{-st}(-t)dt=-\int_0^\infty tf(t)e^{-st}dt$

$=-\mathcal{L}[tf(t)]$

例26. $f(t)=t\sin(at)$

<sol> $\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\underline{\int_0^\infty t\sin(at)e^{-st}dt}$ X

$\mathcal{L}[\sin(at)]=\frac{s}{s^2+a^2}=F(s)$

$\Rightarrow \mathcal{L}[t\sin(at)]=-\frac{d}{ds}F(s)=-\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)=-\frac{-a\cdot 2s}{(s^2+a^2)^2}=\frac{2as}{(s^2+a^2)^2}$ #

例29 $\mathcal{L}[\frac{2}{t}(1-\cosh at)]$ 積分

<sol> $\mathcal{L}[2(1-\cosh at)]=\mathcal{L}[2-2\cosh at]=\frac{2}{s}-\frac{2s}{s^2-a^2}=F(s)$

$\Rightarrow F(u)=\frac{2}{u}-\frac{2u}{u^2-a}$

$\therefore \mathcal{L}[\frac{2}{t}(1-\cosh at)]=\int_s^\infty F(u)du=\int_s^\infty \frac{2}{u}-\frac{2u}{u^2-a^2}du$

$=[\ln|u|^2-\ln|u^2-a^2|]\Big|_{u=s}^{u=\infty}=\ln\left|\frac{u^2}{u^2-a^2}\right|\Big|_{u=s}^{u=\infty}=\ln 1-\ln\left|\frac{s^2}{s^2-a^2}\right|$

$=\ln\left|\frac{s^2-a^2}{s^2}\right|=\ln(1-\frac{a^2}{s^2})$ #

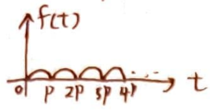
七、週期函數之 Laplace

(一)意義：

滿足 $f(t+p)=f(t)$，稱為週期函數，其中最小正數 P 為 $f(t)$ 之週期

(二)公式：$f(t+p)=f(t) \Rightarrow \mathcal{L}[f(t)]=\dfrac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt$

<p.f.>



$\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt=\int_0^p f(t)e^{-st}dt+\int_p^{2p} f(t)e^{-st}dt+\cdots$

$=\sum_{k=0}^{\infty}\int_{kp}^{(k+1)p} f(t)e^{-st}dt$

令 $z=t-kp \Rightarrow t=z+kp \Rightarrow dt=dz$ 代入

$=\sum_{k=0}^{\infty}\int_0^p f(z+kp)e^{-s(z+kp)}dz=\sum_{k=0}^{\infty}\int_0^p f(z+kp)e^{-sz}\cdot e^{-skp}dz$

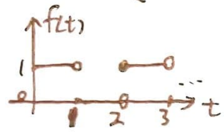
$=\sum_{k=0}^{\infty}e^{-skp}\int_0^p \underbrace{f(z+kp)}_{=f(z)}e^{-sz}dz=(1+e^{-ps}+e^{-2ps}+\cdots)\int_0^p f(z)e^{-sz}dz$

$=\underbrace{\dfrac{1}{1-e^{-ps}}}_{\text{等比級數}}\int_0^p f(z)e^{-sz}dz$ (P為週期)

例31. $\mathcal{L}[g(t)]=G(s)=\dfrac{1}{1-e^{-Ts}}\int_0^T g(t)e^{-st}dt$

$f(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases}$，$f(t+2)=f(t)$

<sol>



週期 $T=2$

$\therefore \mathcal{L}[f(t)]=\dfrac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt=\dfrac{1}{1-e^{-2s}}\int_0^1 1e^{-st}dt=\dfrac{1}{1-e^{-2s}}\left(\dfrac{e^{-st}}{-s}\right)\Big|_{t=0}^{t=1}$

$=\dfrac{1}{1+e^{-2s}}\left(-\dfrac{1}{s}e^{-s}+\dfrac{1}{s}\right)=\dfrac{1-e^{-s}}{s(1-e^{-2s})}=\dfrac{1-e^{-s}}{s[1^2-(e^{-s})^2]}$

$=\dfrac{\cancel{1-e^{-s}}}{s[(1+e^{-s})\cancel{(1-e^{-s})}]}=\dfrac{1}{s(1+e^{-s})}$ #

八、初值定理：

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow f(0)=\lim_{s\to\infty}sF(s)$

九、終值定理：

若 $\mathcal{L}[f(t)]=F(s) \Rightarrow f(\infty)=\lim_{s\to 0} sF(s)$ → 條件：分母根實部須為負

<p.f.> $\mathcal{L}[f'(t)]=sF(s)-f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)\overset{0}{e^{-st}}dt=\lim_{s\to\infty}sF(s)-f(0) \Rightarrow f(0)=\lim_{s\to\infty}sF(s)$

<p.f.> $\mathcal{L}[f'(t)]=sF(s)-f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)\overset{1}{e^{-st}}dt=\lim_{s\to 0}sF(s)-f(0)$

$\Rightarrow \int_0^\infty f'(t)dt=\lim_{s\to 0}sF(s)-f(0) \Rightarrow f(t)\Big|_0^\infty=\lim_{s\to 0}sF(s)-f(0) \Rightarrow f(\infty)-\cancel{f(0)}=\lim_{s\to 0}sF(s)-\cancel{f(0)}$

$\Rightarrow f(\infty)=\lim_{s\to 0}sF(s)$

例32 $\mathcal{L}[f(t)]=\dfrac{3s^2-s+8}{s^3+4s^2-s-4}$ 求 $f(0)=?$

<sol> $f(0)=\lim_{s\to\infty}sF(s) \Rightarrow \lim_{s\to\infty}s\times\dfrac{3s^2-s+8}{s^3+4s^2-s-4}=\lim_{s\to\infty}\dfrac{3s^3-s^2+8s}{1s^3+4s^2-s-4}=3$

例34 $F(s)=\dfrac{1}{s(s^2+2s+2)}$ 求 $\lim_{t\to\infty}f(t)=f(\infty)$

<sol> $\lim_{s\to 0}sF(s)=\lim_{s\to 0}\cancel{s}\cdot\dfrac{1}{\cancel{s}(s^2+2s+2)}=\lim_{s\to 0}\boxed{\dfrac{1}{s^2+2s+2}}=\dfrac{1}{2}$ #

☆ 分母 s^2+2s+2 之根 $s=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-2\pm\sqrt{2^2-4\times1\times2}}{2}=\dfrac{-2\pm\sqrt{-4}}{2}=-1\pm i$ → 實部需 <0

§5-4 Laplace 逆轉換

一、基本型：

(1) $\mathcal{L}^{-1}\left[\frac{1}{s}\right]=1$

(2) $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$

(3) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

(4) $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

(5) $\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$

(6) $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh(at)$

(7) $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh(at)$

$\mathcal{L}[t^n]=\frac{\Gamma^{n+1}}{s^{n+1}}=\frac{n!}{s^{n+1}} \Rightarrow \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right]=t^n \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$ ★ $\begin{cases} \text{ex: } \mathcal{L}^{-1}\left[\frac{1}{s^4}\right]=\frac{t^3}{3!}=\frac{1}{6}t^3 \\ \text{ex: } \mathcal{L}^{-1}\left[\frac{1}{s^5}\right]=\frac{t^4}{4!}=\frac{1}{24}t^4 \end{cases}$

例35. (1) $\mathcal{L}^{-1}\left[\frac{1}{(s+2)(s+3)}\right]$ (2) $\mathcal{L}^{-1}\left[\frac{1}{s^2}+\frac{3s}{s^2+9}-\frac{8}{s^2+16}\right]$

<sol> (1) $\frac{1}{(s+2)(s+3)}=\frac{1}{s+2}+\frac{-1}{s+3} \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+2}-\frac{1}{s+3}\right]=\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]-\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]=e^{-2t}-e^{-3t}$ #

(2) $\mathcal{L}^{-1}\left[\frac{1}{s^2}+\frac{3s}{s^2+9}-\frac{2\times 4}{\underset{s^2+4^2}{s^2+16}}\right]=\frac{t^1}{1!}+\cos 3t-2\sin 4t$

二、s變數平移逆轉換：

$\mathcal{L}[e^{at}f(t)]=F(s-a) \Rightarrow \mathcal{L}^{-1}[F(s-a)]=e^{at}f(t)=e^{at}\mathcal{L}^{-1}[F(s)]$

三、t變數平移逆轉換：

$\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s) \Rightarrow \mathcal{L}^{-1}[e^{-as}F(s)]=f(t-a)u(t-a)$

☆四、微分後的Laplace逆轉換

$\Rightarrow \begin{cases} \frac{d}{ds}F(s)=-\mathcal{L}[tf(t)] \Rightarrow \boxed{\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right]=-tf(t)} \\ \frac{d^2}{ds^2}F(s)=\mathcal{L}[t^2f(t)] \Rightarrow \mathcal{L}^{-1}\left[\frac{d^2}{ds^2}F(s)\right]=t^2f(t) \end{cases}$

$\Rightarrow$ 看到 $\mathcal{L}^{-1}\left[\frac{1}{[F(s)]^2}\right]$ ex: $\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right]$

五、積分後的Laplace逆轉換

$\Rightarrow \begin{cases} \frac{1}{s}F(s)=\mathcal{L}\left[\int_0^t f(z)dz\right] \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t f(z)dz \\ \frac{1}{s^2}F(s)=\mathcal{L}\left[\int_0^t\int_0^y f(z)dz\,du\right] \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^2}F(s)\right]=\int_0^t\int_0^y f(z)dz\,du \end{cases}$

§5-5 摺積定理 (convolution 卷積；摺積)

一、定義：$f(t) \circledast g(t)=\int_0^t f(t-\alpha)g(\alpha)d\alpha$ (背)

convolution

二、定理：

若 $\mathcal{L}[f(t)]=F(s)$，$\mathcal{L}[g(t)]=G(s) \Rightarrow \mathcal{L}[f(t)*g(t)]=\mathcal{L}[f(t)]\mathcal{L}[g(t)]=F(s)\cdot G(s)$

<p.f.> $\mathcal{L}[f(t)\circledast g(t)]=\int_0^\infty\int_0^t f(t-\alpha)g(\alpha)d\alpha\cdot e^{-st}dt=\int_0^\infty\int_0^t f(t-\alpha)g(\alpha)e^{-st}d\alpha\,dt$

$=\int_0^\infty\int_\alpha^\infty f(t-\alpha)g(\alpha)e^{-st}dt\,d\alpha$ 令 $u=t-\alpha \Rightarrow t=u+\alpha \Rightarrow dt=du$

$=\int_0^\infty\int_0^\infty f(u)g(\alpha)e^{-s(u+\alpha)}du\,d\alpha=\int_0^\infty\int_0^\infty f(u)g(\alpha)e^{-su}\cdot e^{-s\alpha}du\,d\alpha$

$=\left(\int_0^\infty f(u)e^{-su}du\right)\left(\int_0^\infty g(\alpha)e^{-s\alpha}d\alpha\right)$

$=\left(\int_0^\infty f(t)e^{-st}dt\right)\left(\int_0^\infty g(t)-e^{-st}dt\right)=F(s)G(s)$

第七章：狀態方程式

• 前言：Linear Time Invariant (線性系統)

裡面的狀態

$$\begin{cases} \dot{x}_{(t)} = A x_{(t)} + B u_{(t)} \quad \cdots\cdots \text{state eq.} \\ y_{(t)} = C x_{(t)} + D u_{(t)} \quad \cdots\cdots \text{o/p e.q.} \end{cases}$$

state eq. → 狀態方程式；o/p e.q. → 輸出方程式；兩者合稱 dynamic e.q. → 動態方程式

狀態指儲能元件(L、C)

取 Laplace Transformer：

$$sX_{(s)} - X_{(0)} = AX_{(s)} + BU_{(s)}$$

$$\Rightarrow (sI-A)X_{(s)} = X_0 + BU_{(s)} \Rightarrow X_{(s)} = (sI-A)^{-1}X_{(0)} + (sI-A)^{-1}BU_{(s)}$$

代入 o/p eq：

$$Y_{(s)} = C\left[(sI-A)^{-1}X_{(0)} + (sI-A)^{-1}BU_{(s)}\right] + DU_{(s)}$$

$$= C(sI-A)^{-1}\underbrace{X_{(0)}}_{\text{初值}} + \left[C(sI-A)^{-1}B + D\right]U_{(s)}$$

得 Transformer function：$H_{(s)}$ 轉移方程式 ☆(設無輸入，只看 L、C 初始儲能)

$$H_{(s)} = \frac{Y_{(s)}}{U_{(s)}} = C(sI-A)^{-1}B + D = C\frac{\operatorname{adj}(sI-A)}{|sI-A|}B + D$$

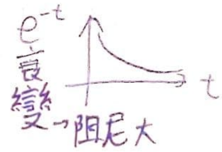
可知 characteristic Equation (C.E.)

為 $|sI-A|=0$ ⇒ 可求得 s，稱為特徵根、pole、Eigen value、自然頻率

[註] 所求得之特徵根(自然 Freq.)：

① 相異實根：(over damping 欠阻尼)

$s=-1, -5 \Rightarrow c_1e^{-t} + c_2e^{-5t}$



② 相同實根：(critical damping 臨界阻尼)

$s=-3, -3 \Rightarrow c_1e^{-3t} + c_2e^{-3t}$

③ 共軛複數根：(過阻尼：under damping)

$s=-2\pm j5 \Rightarrow e^{-2t}(c_1\cos 5t + c_2\sin 5t)$

④ 共軛虛根：(undamping：無阻尼)

$s=\pm j5 \Rightarrow c_1\cos 5t + c_2\sin 5t$

第八章 開關電路

• 前言：

Complete response $\stackrel{工數}{=}$ Natural resp. + Force resp.
$y_n = y_h$ 齊性解　$y_f = y_p$ 特解

$\stackrel{自控}{=}$ Zero state resp. + Zero Input resp.
<暫 + 穩>　<暫態>

$\stackrel{電路}{=}$ Transicont resp. + Steady state resp.
暫態　穩態

自然響應 / 零輸入 / 零態（暫態） ⇒ 齊性解

強迫響應 / 零態（穩態） ⇒ 特解

• 儲能元件

(1) Capacitor

$W_c(0^+) = W_c(0^-)$
$\Rightarrow \frac{1}{2}CV_c^2(0^+) = \frac{1}{2}CV_c^2(0^-)$

☆ $V_c(0^+) = V_c(0^-)$ Cap. 的 Voltage 不會瞬間變化

[例外] ① 若某迴路僅有 Cap 與 Voltage Source，則 $V_c(0^+) \neq V_c(0^-)$

② 若 Voltage Source 為 $\delta(t)$，則 $V_c(0^+) \neq V_c(0^-)$；處理方法用 Laplace，再用初值 theorem

(2) Inductor

$W_L(0^+) = W_L(0^-)$
$\Rightarrow \frac{1}{2}Li_L^2(0^+) = \frac{1}{2}Li_L^2(0^-)$
$\Rightarrow i_L(0^+) = i_L(0^-)$

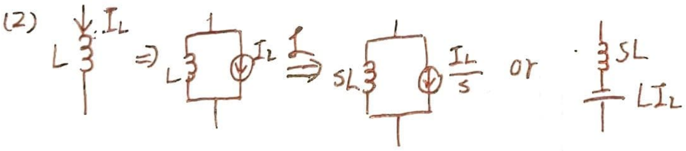
☆ 電感器之 I 不會瞬間變化

[例外] ① 若某 node 僅有 L 與 I source，則 $i_L(0^+) \neq i_L(0^-)$

② 若 Voltage Source 為 $\delta(t)$，$i_L(0^+) \neq i_L(0^-)$，處理方法：用 L.T.，再用初值 thm。

[補充] (1)

(2)



[總結] 若 Voltage Source 為 DC，則：